INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 1.14]

MATRIX EQUATIONS – THE LINEAR SYSTEM AX=B

Compiled by Christos Nikolaidis

A. Paper 1 questions (SHORT)

MATRIX EQUATIONS

1. [Maximum mark: 6]

Let $A=\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$, $B=\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}$.

Find a matrix X such that $2A+3X=B$ by using two methods:

(i) **Method A**: Assume that $X=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ and find a,b,c,d .

(ii) **Method B**: Solve the equation for X and then calculate X .

....

....

....

....

....

....

....

....

....

....

....

....

....

....

2. [Maximum mark: 8]

Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, $B=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$.

(a) Find A^{-1}. [1]

(b) Find a matrix X such that $AX=B$ by using two methods:

(i) **Method A**: Assume that $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and find a,b,c,d.

(ii) **Method B**: Solve the equation for X and then calculate X. [5]

(c) Find a matrix X such that $XA=B$ [use method B only]. [2]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

3. [Maximum mark: 7]

All matrices in this question are 2×2 matrices. Provided that matrix $\boldsymbol{A}$ has an inverse, solve the following matrix equations

Equation	**Solve for $\boldsymbol{X}$**
$\boldsymbol{X} - \boldsymbol{A} = \boldsymbol{B}$	
$\boldsymbol{A} + 2\boldsymbol{X} = \boldsymbol{B}$	
$\boldsymbol{AX} = \boldsymbol{C}$	
$\boldsymbol{XA} = \boldsymbol{C}$	
$\boldsymbol{AXA} = \boldsymbol{C}$	
$\boldsymbol{AX} - \boldsymbol{B} = \boldsymbol{C}$	
$2\boldsymbol{AX} + 3\boldsymbol{B} = \boldsymbol{C}$	

4. [Maximum mark: 6]

The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{X}$ are all invertible 3 × 3 matrices.

(a) Given that $\boldsymbol{A}^{-1}\boldsymbol{XB} = \boldsymbol{C}$, express $\boldsymbol{X}$ in terms of the other matrices. [2]

(b) Provided that $\boldsymbol{A} - \boldsymbol{I}$ has an inverse, solve the matrix equations

(i) $\boldsymbol{AX} - \boldsymbol{X} = \boldsymbol{C}$ (ii) $\boldsymbol{XA} - \boldsymbol{X} = \boldsymbol{C}.$ [4]

....

....

....

....

....

....

....

....

....

5. [Maximum mark: 4]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$.

(a) Find $\boldsymbol{A}^2$. [1]

(b) Let $\boldsymbol{B} = \begin{pmatrix} -3 & 4 \\ 2 & 1 \end{pmatrix}$. Solve the matrix equation $3\boldsymbol{X} + \boldsymbol{A} = \boldsymbol{B}$. [3]

....

....

....

....

....

....

....

....

....

6. [Maximum mark: 5]

Given that $\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ find $\boldsymbol{X}$ if $\boldsymbol{BX} = \boldsymbol{A} - \boldsymbol{AB}$.

....

....

....

....

....

....

....

....

....

....

....

....

7. [Maximum mark: 4]

A and B are 2 × 2 matrices, where $A = \begin{bmatrix} 5 & 2 \\ 2 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 11 & 2 \\ 44 & 8 \end{bmatrix}$. Find B.

....

....

....

....

....

....

....

....

....

....

....

8. [Maximum mark: 4]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$.

(a) Find $\boldsymbol{AB}$. [1]

(b) Solve $\boldsymbol{A}^{-1}\boldsymbol{X} = \boldsymbol{B}$. [3]

....

....

....

....

....

....

....

....

....

....

....

9. [Maximum mark: 6]

Let $\boldsymbol{A} = \begin{pmatrix} 5 & 1 \\ 6 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & -1 \\ -6 & 5 \end{pmatrix}$.

(a) (i) Find $\boldsymbol{AB}$. (ii) Write down the inverse of $\boldsymbol{A}$. [3]

Let $\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$.

(b) Solve the matrix equation $\boldsymbol{AX} = \boldsymbol{C}$. [3]

....

....

....

....

....

....

....

....

10. [Maximum mark: 6]

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 5 & -2 \\ 7 & 1 \end{pmatrix}$.

(a) Write down the inverse, $\boldsymbol{A}^{-1}$. [2]

(b) $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{X}$ are also 2 × 2 matrices.

(i) Given that $\boldsymbol{XA} + \boldsymbol{B} = \boldsymbol{C}$, express $\boldsymbol{X}$ in terms of $\boldsymbol{A}^{-1}$, $\boldsymbol{B}$ and $\boldsymbol{C}$.

(ii) Given that $\boldsymbol{B} = \begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix}$, find $\boldsymbol{X}$. [4]

....

....

....

....

....

....

....

....

11. [Maximum mark: 4]

Let $\boldsymbol{A} = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$.

(i) Find $\boldsymbol{A}^{-1}$. (ii) Solve the matrix equation $\boldsymbol{AX} = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix}$.

....

....

....

....

....

....

....

....

12. [Maximum mark: 7]

Matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are defined by $\boldsymbol{A} = \begin{pmatrix} 5 & 1 \\ 7 & 2 \end{pmatrix}$ $\boldsymbol{B} = \begin{pmatrix} 2 & 4 \\ -3 & 15 \end{pmatrix}$ $\boldsymbol{C} = \begin{pmatrix} 9 & -7 \\ 8 & 2 \end{pmatrix}$.

Let $\boldsymbol{X}$ be an unknown 2 × 2 matrix satisfying the equation $\boldsymbol{AX} + \boldsymbol{B} = \boldsymbol{C}$.

This equation may be solved for $\boldsymbol{X}$ by rewriting it in the form $\boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{D}$, where $\boldsymbol{D}$ is a 2×2 matrix.

(a) Write down $\boldsymbol{A}^{-1}$. [2]

(b) Find $\boldsymbol{D}$. [3]

(c) Find $\boldsymbol{X}$. [2]

....

....

....

....

....

....

....

....

....

13. [Maximum mark: 6]

The matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{X}$ are given by

$\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ -5 & 6 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix}$, $\boldsymbol{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Q}$.

Given that $\boldsymbol{AX} + \boldsymbol{X} = \boldsymbol{B}$, find the **exact** values of a, b, c and d.

....

....

....

....

....

....

....

....

....

....

....

....

14. [Maximum mark: 5]

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -3 & 0 \\ 4 & -2 & 1 \end{pmatrix}$.

(a) Write down $\boldsymbol{A}^{-1}$. [1]

(b) Let $\boldsymbol{B}$ be a 3 × 3 matrix. Given that $\boldsymbol{AB} + \begin{pmatrix} -3 & 2 & 1 \\ 5 & 3 & 4 \\ -9 & 2 & 10 \end{pmatrix} = \begin{pmatrix} 7 & 6 & -7 \\ 6 & 5 & -8 \\ 1 & 7 & -5 \end{pmatrix}$, find $\boldsymbol{B}$. [4]

....

....

....

....

....

....

THE LINEAR SYSTEM $AX = B$

15. [Maximum mark: 6]

Let $A = \begin{pmatrix} 7 & 8 \\ 2 & 3 \end{pmatrix}$

(a) by using the appropriate formulas, find

(i) $\det A$

(ii) A^{-1}. [3]

(b) **Hence**, solve the system of simultaneous equations.

$$7x + 8y = 1$$
$$2x + 3y = 1$$ [3]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

16. [Maximum mark: 8]

Let $\boldsymbol{M} = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$.

(a) Write down the determinant of $\boldsymbol{M}$. [1]

(b) Write down $\boldsymbol{M}^{-1}$ [1]

(b) **Hence** solve $\boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$. [3]

(c) Write down

(i) a system of equations represented by the matrix equation in (c).

(ii) the solution of the system [3]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

17. [Maximum mark: 6]

A matrix $\boldsymbol{M}$ has **inverse** $\boldsymbol{M}^{-1} = \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}$. Let $\boldsymbol{B} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ and $\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$.

(a) Find $\boldsymbol{M}$ [3]

(b) Solve the matrix equation $\boldsymbol{MX} = \boldsymbol{B}$ [3]

....

....

....

....

....

....

....

....

....

....

18. [Maximum mark: 5]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & -1 & 4 \\ 2 & 4 & -3 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

(a) Write down $\boldsymbol{A}^{-1}$. [2]

(b) Solve $\boldsymbol{AX} = \boldsymbol{B}$. [3]

....

....

....

....

....

....

....

....

....

....

19. [Maximum mark: 6]

(a) Write down the inverse of the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 1 & -5 & 3 \end{pmatrix}$. [2]

(b) **Hence** solve the simultaneous equations

$$\begin{aligned} x - 3y + z &= 1 \\ 2x + 2y - z &= 2 \\ x - 5y + 3z &= 3 \end{aligned}$$ [4]

....

....

....

....

....

....

....

....

....

20. [Maximum mark: 6]

(a) Write down the inverse of the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}$ [2]

(b) **Hence** solve

$$\begin{aligned} x - 3y &= 1 \\ 2x + z &= 2 \\ 4x + y + 3z &= -1 \end{aligned}$$ [4]

....

....

....

....

....

....

....

....

21. [Maximum mark: 6]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 18 \\ 23 \\ 13 \end{pmatrix}$ and $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(a) Write down the inverse matrix $\boldsymbol{A}^{-1}$.

(b) Consider the equation $\boldsymbol{AX} = \boldsymbol{B}$.

(i) Express $\boldsymbol{X}$ in terms of $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$. (ii) **Hence**, solve for $\boldsymbol{X}$.

....

....

....

....

....

....

....

....

....

22. [Maximum mark: 6]

The system of linear equations below can be written as the matrix equation $\boldsymbol{MX} = \boldsymbol{N}$.

$$\begin{aligned} x + 6y - 3z &= -1 \\ 4x + 2y - 4z &= 12 \\ x + y + 5z &= 15 \end{aligned}$$

(a) Write down the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$. [2]

(b) Solve the **matrix** equation $\boldsymbol{MX} = \boldsymbol{N}$. [3]

(c) Hence write down the solution of the system of linear equations. [1]

....

....

....

....

....

....

....

....

23. [Maximum mark: 6]

The matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & -1 \\ 2 & -2 & 1 \end{pmatrix}$ has inverse $\boldsymbol{A}^{-1} = \begin{pmatrix} -1 & -2 & -2 \\ 3 & 1 & 1 \\ a & 6 & b \end{pmatrix}$.

(a) Use the fact $\boldsymbol{AA}^{-1} = \boldsymbol{I}$ to find the value of (i) a; (ii) b. [2]

Consider the simultaneous equations

$$x + 2y = 7$$
$$-3x + y - z = 10$$
$$2x - 2y + z = -12$$

(b) Write these equations as a matrix equation. [1]

(c) Solve the matrix equation. [3]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

B. Paper 2 questions (*LONG*)

24. [Maximum mark: 11]

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

(a) Find (i) $\boldsymbol{A}^{-1}$; (ii) $\boldsymbol{A}^{2}$. [4]

Let $\boldsymbol{B} = \begin{pmatrix} p & 2 \\ 0 & q \end{pmatrix}$.

(b) Given that $2\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 2 & 6 \\ 4 & 3 \end{pmatrix}$, find the value of p and of q. [3]

(c) Hence find $\boldsymbol{A}^{-1}\boldsymbol{B}$. [2]

(d) Let $\boldsymbol{X}$ be a 2×2 matrix such that $\boldsymbol{AX} = \boldsymbol{B}$. Find $\boldsymbol{X}$. [2]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

25. [Maximum mark: 12]

Let $\boldsymbol{M} = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}$, where $a \in \mathbb{Z}$.

(a) Find $\boldsymbol{M}^2$ in terms of a [4]

(b) If $\boldsymbol{M}^2$ is equal to $\begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$, find the value of a. [2]

(c) Using this value of a, find $\boldsymbol{M}^{-1}$ and **hence** solve the system of equations:

$$-x + 2y = -3$$
$$2x - y = 3$$ [6]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

26. [Maximum mark: 13]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$.

(a) Write down $\boldsymbol{A}^{-1}$. [2]

The matrix $\boldsymbol{B}$ satisfies the equation $\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{B}\right)^{-1} = \boldsymbol{A}$, where $\boldsymbol{I}$ is the 3×3 identity matrix.

(b) (i) Show that $\boldsymbol{B} = -2(\boldsymbol{A}^{-1} - \boldsymbol{I})$ and **hence** find $\boldsymbol{B}$.

(ii) Write down $\det \boldsymbol{B}$ and h**ence**, explain why $\boldsymbol{B}^{-1}$ exists. [6]

Let $\boldsymbol{BX} = \boldsymbol{C}$, where $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

(c) (i) Find $\boldsymbol{X}$.

(ii) Write down a system of equations whose solution is represented by $\boldsymbol{X}$. [5]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

27. [Maximum mark: 15]

Let $f(x)=ax^2+bx+c$ where a, b and c are rational numbers.

(a) The point P(–4, 3) lies on the curve of f Show that $16a-4b+c=3$. [2]

(b) The points Q(6, 3) and R(–2, –1) also lie on the curve of f. Write down two other linear equations in a, b and c. [2]

(c) These three equations may be written as a matrix equation in the form $\boldsymbol{AX}=\boldsymbol{B}$, where $\boldsymbol{X}=\begin{pmatrix}a\\b\\c\end{pmatrix}$.

(i) Write down the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

(ii) Write down $\boldsymbol{A}^{-1}$.

(iii) **Hence** or otherwise, find $f(x)$. [8]

(d) Write $f(x)$ In the form $f(x)=a(x-h)^2+k$, where a, h and k are rational numbers. [3]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

28. [Maximum mark: 14]

The function f is given by $f(x) = mx^3 + nx^2 + px + q$, where m, n, p, q are integers.

The graph of f passes through the points (0, 0) and (3, 18).

(a) Write down the value of q. [1]

(b) Show that $27m + 9nx + 3p = 18$. [2]

The graph of f also passes through the points (1, 0) and (–1, –10).

(c) Write down the other two linear equations in m, n and p. [2]

(d) (i) Write down these three equations as a matrix equation.

(ii) Solve this matrix equation. [6]

(e) The function f can also be written $f(x) = x(x-1(rx-s)$ where r and s are integers. Find r and s. [3]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....